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# Vortices, instantons and branes

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Abstract: The purpose of this paper is to describe a relationship between the moduli space of vortices and the moduli space of instantons. We study charge $k$ vortices in $\mathrm{U}(N)$ Yang-Mills-Higgs theories and show that the moduli space is isomorphic to a special lagrangian submanifold of the moduli space of $k$ instantons in non-commutative $\mathrm{U}(N)$ Yang-Mills theories. This submanifold is the fixed point set of a $\mathrm{U}(1)$ action on the instanton moduli space which rotates the instantons in a plane. To derive this relationship, we present a D-brane construction in which the dynamics of vortices is described by the Higgs branch of a $\mathrm{U}(k)$ gauge theory with 4 supercharges which is a truncation of the familiar ADHM gauge theory. We further describe a moduli space construction for semi-local vortices, lumps in the $\mathbf{CP}^N$ and grassmannian sigma-models, and vortices on the non-commutative plane. We argue that this relationship between vortices and instantons underlies many of the quantitative similarities between quantum field theories in two and four dimensions.

Keywords: Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories.



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## Contents



## 1. Introduction and preview

The moduli space of a supersymmetric system is defined as the set of classically massless, or light, degrees of freedom. The beauty of this concept lies in the fact that much of the low-energy behaviour of the system may be encoded as geometrical features on the moduli space. Whether the subject be string compactifications, the dynamics of gauge theories, or the interactions of solitons, the moduli space approximation provides an effective, and tractable, approach to extract the infra-red quantum properties of the system.

One particularly useful geometric feature of the moduli space is the metric, describing the kinetic interactions of the system. Our interest in this paper will be focused on the moduli space of solitons, specifically vortices. In this case, the relevance of the metric was first revealed by Manton who showed that geodesics on the moduli space track the classical scattering of solitons [1].

It is common lore that for dynamics exhibiting 8 or more supercharges, the metric on the moduli space is exactly calculable. For theories with 4 supercharges or less, the metric can, in general, only be computed in asymptotic regimes. In the context of solitons, both

Yang-Mills instantons and monopoles preserve up to 8 supercharges and indeed exact, albeit somewhat implicit, expressions for the metrics are known using the techniques of [2, 3].[1] In contrast, vortices preserve a maximum of only 4 supercharges, and knowledge of the metric is currently restricted to the situation where the solitons are well-separated [5, 6].

Nevertheless, it is possible to make progress in supersymmetric quantum field theories even when the moduli space metric is not known. This is because, as first emphasised by Witten [7], many of the simplest quantities of interest depend only on topological characteristics of the moduli space. For example, the supersymmetric bound states of solitons are related to various cohomology classes of the moduli space [7, 8].[2] Similarly non-perturbative contributions to BPS correlation functions, which involve integrals over the moduli space of instantons, often reduce to topological invariants [9, 10]. Thus, for many purposes it suffices to know only crude topological information about the moduli space.

The purpose of this paper is to describe the moduli space of vortices in $\mathrm{U}(N)$ Yang-Mills-Higgs theories where the gauge group is broken completely by $N$ fundamental scalar fields. The theory has a mass gap and exhibits vortices, labeled by the winding number $k$ of the magnetic field,

$$\mathrm{Tr}\int B = -2\pi k\,. \tag{1.1}$$

Here we summarise our main results. We start in section 2 with a study of the moduli space of charge $k>0$ vortices which we shall denote as $\mathcal{V}_{k,N}$. Our first result concerns the real dimension of the moduli space which, using index theory techniques, we show to be

$$\dim(\mathcal{V}_{k,N}) = 2kN\,. \tag{1.2}$$

In section 3 we present a brane construction of the vortices, from which we extract a description of $\mathcal{V}_{k,N}$ as a $\mathrm{U}(k)$ symplectic quotient of $\mathbf{C}^{k(N+k)}$. This quotient construction is most easily described as the Higgs branch of a $\mathrm{U}(k)$ gauge theory with four supercharges, coupled to a single adjoint chiral multiplet and $N$ fundamental chiral multiplets.

The moduli space $\mathcal{V}_{k,N}$ naturally inherits a metric from the Kähler quotient construction. This does not agree with the Manton metric describing the classical scattering of solitons. Given our discussion above, this is neither unexpected nor an obstacle to utilising our construction for further calculations. As we shall see, the inherited metric is a deformation of the Manton metric, preserving the Kähler property, the isometries and the asymptotic form.

The parametric scaling of the dimension (1.2) is reminiscent of the moduli space of $k$ instantons in a $\mathrm{U}(N)$ gauge theory, which we shall denote as $\mathcal{I}_{k,N}$. Recall that the real dimension of the instanton moduli space is

$$\dim(\mathcal{I}_{k,N}) = 4kN\,.$$

Moreover, those familiar with instanton moduli spaces will have recognised the quotient

[1] For particularly simple cases, more explicit descriptions also exist. See, for example [4].

[2] Questions of $L^2$ normalisability mean that asymptotic behaviour of the metric is also required.

construction of $\mathcal{V}_{k,N}$ as a truncated version of the ADHM quotient [2]. In section 4, we make this relationship more explicit and show that the moduli space of vortices $\mathcal{V}_{k,N}$ is a complex middle-dimensional submanifold (or, since $\mathcal{I}_{k,N}$ is hyperKähler, equivalently a special lagrangian submanifold) of the resolved instanton moduli space $\mathcal{I}_{k,N}$. We further show that $\mathcal{V}_{k,N}$ may be realised as the fixed point set of a holomorphic U(1) action on $\mathcal{I}_{k,N}$, descending from the rotations of instantons in a plane.

In section 5, we generalise this construction to vortices in $\mathrm{U}(N)$ gauge theories with $N_f = N + M$ flavours. For abelian gauge theories, such objects have been well studied and are known as semi-local vortices. In the strong coupling limit these vortices become lump solutions on the $\mathbf{CP}^M$ Higgs branch of the theory. For the non-abelian theory, these vortices are related to lumps in the $G(N, N_f)$ grassmannian sigma-model of $N$ planes in $\mathbf{C}^{N_f}$. We denote these moduli spaces of vortices as $\hat{\mathcal{V}}_{k,(N,M)}$ (note that $\mathcal{V}_{k,N} \cong \hat{\mathcal{V}}_{k,(N,0)}$). The dimension of the moduli space is,

$$\dim(\hat{\mathcal{V}}_{k,(N,M)}) = 2k(N + M)\,.$$

We again give a brane construction as well as a quotient construction of the moduli space and explain how it can be described as the fixed point set of a (different) holomorphic action on the moduli space of instantons $\mathcal{I}_{k,N_f}$.

In section 6, we consider the Yang-Mills-Higgs theory defined on the spatial non-commutative plane with $[x^1, x^2] = -i\vartheta$. We describe how the moduli space of vortices changes as $\vartheta$ is varied. We show that the moduli spaces may become singular, cease to exist, or undergo interesting topology changing transitions for different values of $\vartheta$. We end in section 7 with conclusions and a discussion.

## 2. Vortices

Our starting point is the maximally supersymmetric theory admitting vortex solutions which, for concreteness, we choose to live in $d = 2 + 1$ dimensions with $\mathcal{N} = 4$ supersymmetry.[3] Our theory includes a $\mathrm{U}(N)$ vector multiplet, consisting of a gauge field $A_\mu$, a triplet of adjoint scalar fields $\phi^r$, $r = 1, 2, 3$ and their fermionic partners. To these we couple $N$ fundamental hypermultiplets, each of which contains two complex scalars $q$ and $\tilde{q}$, and their partner fermions. As well as the $\mathrm{U}(N)_G$ gauge symmetry, the lagrangian also enjoys a $\mathrm{SU}(N)_F$ flavour symmetry. Under these two groups, the $q$ field transforms as $(\mathbf{N}, \bar{\mathbf{N}})$, while $\tilde{q}$ transforms as $(\bar{\mathbf{N}}, \mathbf{N})$. In the following we take both $q$ and $\tilde{q}$ to represent $N \times N$ matrices,

$$q = q^a_{\ i}\,, \qquad \tilde{q} = \tilde{q}^i_{\ a} \qquad a, i = 1, \dots, N\,,$$

where the $a$ index furnishes a representation under $\mathrm{U}(N)_G$ while the $i$ index refers to

[3] Supersymmetric theories which admit vortices exist in any dimension between 1+1 and 5+1. The discussion of quantum effects, particular to each case, is very interesting but will be left for future work.



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$SU(N)_F$. In this notation, the bosonic part of the lagrangian reads,[4]

$$\mathcal{L} = -\operatorname{Tr}\left[\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2e^2}\mathcal{D}\phi^r\mathcal{D}\phi^r + \mathcal{D}_\mu q^\dagger\mathcal{D}^\mu q + \mathcal{D}_\mu\tilde{q}\mathcal{D}^\mu\tilde{q}^\dagger + e^2|q\tilde{q}|^2 + \right.$$
$$\left. + \frac{1}{2e^2}[\phi^r,\phi^s]^2 + (\tilde{q}^\dagger\tilde{q} + qq^\dagger)\phi^r\phi^r + \frac{e^2}{2}\left(qq^\dagger - \tilde{q}^\dagger\tilde{q} - \zeta\mathbf{1}_N\right)^2\right]. \qquad (2.1)$$

The final term in the lagrangian is a D-term and includes a Fayet-Iliopoulos parameter $\zeta$, which we take to be strictly positive $\zeta > 0$. The presence of this parameter induces symmetry breaking with the unique vacuum, up to Weyl permutations, given by

$$q^a{}_i = \sqrt{\zeta}\delta^a{}_i \qquad \tilde{q}^i{}_a = 0\,, \qquad \phi^r = 0\,.$$

The ground state of the theory is a gapped, colour-flavour locking phase with the symmetry breaking pattern

$$U(N)_G \times SU(N)_F \to SU(N)_{\mathrm{diag}} \qquad (2.2)$$

The breaking of the overall $U(1)_G$ gauge symmetry ensures the existence of vortex solutions in the theory. These vortices obey a Bogomoln'yi bound which is the natural generalisation of the usual abelian vortex bound [11] and may be simply determined by the standard trick of completing the square in the hamiltonian. It will turn out that the most general vortex solutions involve only the fields $q$ and $B = F_{12}$, and we choose to set the remaining fields to zero at this stage. Restricting to time independent configurations, the hamiltonian reads,

$$\begin{aligned}\mathcal{H} &= \operatorname{Tr}\left[\frac{1}{2e^2}B^2 + |\mathcal{D}_1 q|^2 + |\mathcal{D}_2 q|^2 + \frac{e^2}{2}(qq^\dagger - \zeta\mathbf{1})\right] \\ &= \operatorname{Tr}\left[\frac{1}{2e^2}(B \mp e^2(qq^\dagger - \zeta\mathbf{1}))^2 + |\mathcal{D}_1 q^\dagger \pm i\mathcal{D}_2 q^\dagger|^2 \mp \zeta B\right] \\ &\geq 2\pi\zeta|k|\,,\end{aligned}$$

where $k \in \mathbb{Z}$ is the winding number of the configuration defined in (1.1). Choosing $k > 0$, the bound is saturated by configurations satisfying the first order Bogomoln'yi equations which, for once, we write with all indices explicit to emphasise their matrix nature

$$\begin{aligned}B^a{}_b &= e^2(q^a{}_i q^{i\dagger}{}_b - \zeta\delta^a{}_b) \\ \mathcal{D}_z q^a{}_i &= \partial_z q^a{}_i - i(A_z)^a{}_b q^b{}_i = 0\,,\end{aligned} \qquad (2.3)$$

where we have introduced the complex coordinate on the spatial plane $z = x^1 + ix^2$.

The main purpose of this paper will be to study the moduli space of solutions to these equations. We denote the moduli space of charge $k$ vortices in the $U(N)$ Yang-Mills-Higgs theory as $\mathcal{V}_{k,N}$. We start with a study of the linearised equations to determine the dimension of $\mathcal{V}_{k,N}$. The reader uninterested in the details of the index theorem may skip to the following subsection where basic properties of $\mathcal{V}_{k,N}$ are discussed, taking with them the following punchline:

$$\dim(\mathcal{V}_{k,N}) = 2kN\,. \qquad (2.4)$$

[4]Our conventions: we choose a hermitean connection with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ and $\mathcal{D}_\mu q = \partial_\mu q - iA_\mu q$. All gauge and flavour indices are implicit and assumed summed, with the exception of $r, s = 1, 2, 3$ which is explicit and summed.



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### 2.1 An index theorem

In this section, we prove the result (2.4) by studying the fluctuations $(\dot{A}, \dot{q})$ around a given solution. Our method follows closely the work of E. Weinberg [12] who analysed the moduli space in the abelian case $N = 1$. The linearised Bogomoln'yi matrix equations are

$$\begin{aligned} \mathcal{D}^a_z \dot{A}_{\bar{z}} - \mathcal{D}^a_{\bar{z}} \dot{A}_z &= \frac{ie^2}{2}\left(\dot{q}q^\dagger + q\dot{q}^\dagger\right) \\ \mathcal{D}^f_z \dot{q} &= i\dot{A}_z q \end{aligned} \tag{2.5}$$

and are to be augmented with a gauge fixing condition, for which we choose Gauss' law,

$$\mathcal{D}^a \dot{A}_{\bar{z}} + \mathcal{D}^a_{\bar{z}} \dot{A}_z = -\frac{ie^2}{2}(\dot{q}q^\dagger - q\dot{q}^\dagger) \tag{2.6}$$

which can be combined with the first of the equations in (2.5) to give

$$\mathcal{D}^a_{\bar{z}} \dot{A}_z = -\frac{ie^2}{2}\dot{q}q^\dagger\,. \tag{2.7}$$

The observant reader will have noticed the appearance of superscripts on the covariant derivatives, which are there to remind us of the $\mathrm{U}(N)_G$ representation of the field on which they act:

$$\mathcal{D}^a X = \partial X - i[A, X]\,, \qquad \mathcal{D}^f Y = \partial X - iAY\,, \qquad \mathcal{D}^{\bar{f}} Y^\dagger = \partial Y^\dagger + iY^\dagger A\,.$$

Before proceeding, notice that it is possible to rescale the gauge field $A \to A/e$ and coordinate $z \to ez$ to remove $e^2$ from the equations. The number of zero modes is therefore independent of $e^2$ and we use this freedom to set $e^2 = 2$ which simplifies the linearised Bogomoln'yi equations somewhat so they can be written as,

$$\Delta\eta \equiv \begin{pmatrix} i\mathcal{D}^a_{\bar{z}} & -q^{r\dagger} \\ q^r & i\mathcal{D}^f_z \end{pmatrix} \begin{pmatrix} \dot{A}_z \\ \dot{q} \end{pmatrix} = 0\,,$$

where the superscript in $q^r$ denotes the fact that the matrix $q$ acts as right multiplication. We now define the index of $\Delta$ as

$$\mathcal{I} = \lim_{M^2\to 0} \mathcal{I}(M^2) \equiv \lim_{M^2\to 0}\left[\mathrm{Tr}\left(\frac{M^2}{\Delta^\dagger\Delta + M^2}\right) - \mathrm{Tr}\left(\frac{M^2}{\Delta\Delta^\dagger + M^2}\right)\right]$$

which counts the number of complex zero modes of $\Delta$ minus the number of zero modes of $\Delta^\dagger$. Let us firstly show that $\Delta^\dagger$ is strictly positive definite, and therefore admits no zero modes, by examining the norm squared of a putative zero mode

$$\begin{aligned} \left|\Delta^\dagger \begin{pmatrix} X \\ Y \end{pmatrix}\right|^2 &= \left|i\mathcal{D}^a_z X + Yq^\dagger\right|^2 + \left|Xq - i\mathcal{D}^f_{\bar{z}} Y\right|^2 \\ &= |\mathcal{D}^a_z X|^2 + \left|\mathcal{D}^f_{\bar{z}} Y\right|^2 + \left|Yq^\dagger\right|^2 + |Xq|^2 = 0\,, \end{aligned}$$

where the vanishing of the cross-terms occurs when evaluated on a solution to (2.3). With all terms on the right-hand side positive definite, the last two terms ensure that $X = Y = 0$.

Thus $\Delta^\dagger$ admits no zero modes and $\mathcal{J}$ counts the number of zero modes of $\Delta$. We now turn to the task of evaluating $\mathcal{J}$. For theories in which the fields have suitable fall-off at spatial infinity (faster than $1/r$ in our case — see the second reference in [12]), the quantity $\mathcal{J}(M^2)$ is independent of $M^2$ and the index $\mathcal{J}$ may be computed more simply in the opposite limit $M^2 \to \infty$. It is a simple matter to derive an explicit expression for the two composite operators,

$$\begin{aligned}\Delta^\dagger\Delta &= -\partial_z\partial_{\bar z}\mathbf{1}_2 + \begin{pmatrix}\Gamma_1 - \frac{1}{2}B^a & L_1 \\ L_2 & \Gamma_2 + \frac{1}{2}B^l\end{pmatrix} \\ \Delta\Delta^\dagger &= -\partial_z\partial_{\bar z}\mathbf{1}_2 + \begin{pmatrix}\Gamma_1 & 0 \\ 0 & \Gamma_2\end{pmatrix},\end{aligned}$$

where the various operators are defined as,

$$\begin{aligned}\Gamma_1 X &= i[\partial_{\bar z}A_z, X] + i[A_z, \partial_{\bar z}X] + i[A_{\bar z}, \partial_z X] - [A_{\bar z}, [A_z, X]] + Xqq^\dagger \\ \Gamma_2 Y &= iA_{\bar z}\partial_z Y + i(\partial_{\bar z}A_z)Y + iA_z\partial_{\bar z}Y + A_{\bar z}A_z Y + Yq^\dagger q \\ L_1 Y &= -iY\mathcal{D}_z^{\bar f}q^\dagger \\ L_2 X &= iX\mathcal{D}_{\bar z}^f q\,.\end{aligned}$$

Expanding $\mathcal{J}(M^2)$ in terms of $(\nabla + M^2) = (-\partial_z\partial_{\bar z} + M^2)$, we have

$$\mathcal{J}(M^2) = -M^2\,\mathrm{Tr}\left[\frac{1}{\nabla + M^2}\begin{pmatrix}-\frac{1}{2}B^a & L_1 \\ L_2 & \frac{1}{2}B^l\end{pmatrix}\frac{1}{\nabla + M^2} + \cdots\right],$$

where the $\ldots$ vanish in the $M^2 \to \infty$ limit. Taking the trace over the adjoint action of $B^a$ causes this term to vanish, and we are left only with the left action of $B$ on the space of $N \times N$ matrices $q$. We thus have,

$$\begin{aligned}\mathcal{J} &= \lim_{M^2\to\infty}\sum_{i=1}^{N}\mathrm{Tr}\int d^2x\int\frac{d^2k}{(2\pi)^2}\frac{-M^2}{(\frac{1}{4}k^2 + M^2)^2}\frac{1}{2}B \\ &= -\sum_{i=1}^{N}\mathrm{Tr}\int\frac{d^2x}{2\pi}B \\ &= Nk\end{aligned}$$

which counts the complex dimension of $\mathcal{V}_{k,N}$ to give the promised result.

### 2.2 The structure of the vortex moduli space

Let us now discuss a few basic facts about the vortex moduli space. On general grounds, the space decomposes as,

$$\mathcal{V}_{k,N} = \mathbf{C} \times \tilde{\mathcal{V}}_{k,N}\,,$$

where $\mathbf{C}$ parameterises the center of mass of the vortex configuration, while information about the relative and internal vortex motion is contained within the $2(kN-1)$-dimensional centered vortex moduli space $\tilde{\mathcal{V}}_{k,N}$. Supersymmetry, and the BPS-nature of the vortices, ensures that the moduli space admits a natural Kähler metric defined by the overlap of the zero modes,

$$\mathcal{L} = \mathrm{Tr}\int d^2x\ \frac{2}{e^2}\dot{A}_z\dot{A}_{\bar{z}} + \dot{q}\dot{q}^\dagger = g_{a\bar{b}}\dot{z}^a\dot{\bar{z}}^{\bar{b}}\,, \tag{2.8}$$

where $z^a$ are complex coordinates on $\mathcal{V}_{k,N}$. This is the Manton metric, descending from the kinetic terms of the lagrangian (2.1) and is such that geodesics of $g$ describe the classical scattering of vortices [1].

For the case of the abelian-Higgs model, $N = 1$, many properties of the vortices and the metric have been studied. Taubes showed long ago that, as expected, the collective coordinates of $\mathcal{V}_{k,N}$ correspond to the positions of $k$ unit charge vortices moving on the plane and may be identified with the zeros of the Higgs field $q$ [13]. The metric on $\mathcal{V}_{k,1}$ can be shown to be geodesically complete and although the exact form of the metric remains unknown for $k \geq 2$, several interesting properties were uncovered by Samols [5]. Asymptotically, the metric approaches the flat metric on $\mathbf{C}^k/S_k$ where $S_k$ is the permutation group of $k$ elements, reflecting the fact that the vortices are indistinguishable particles. The interactions of the vortices resolve the orbifold singularities of $\mathbf{C}^k/S_k$ as the cores overlap. The leading order corrections to the flat metric, which are exponentially suppressed in the separation between vortices, were recently calculated by Manton and Speight [6].

The moduli space of vortices in the non-abelian Yang-Mills-Higgs model does not appear to have been studied in the literature. Here we make a few elementary remarks. The dimension $\dim(\mathcal{V}_{k,N}) = 2kN$ suggests that the charge $k$ vortex again decomposes into $k$ unit charge vortices, each of which is alloted a position on the plane together with $(N-1)$ complex internal degrees of freedom describing the orientation of the vortex in the $\mathrm{SU}(N)_{\mathrm{diag}}$ group. Indeed the action $\mathrm{SU}(N)_{\mathrm{diag}}$ on the fields descends to a natural action on $\tilde{\mathcal{V}}_{k,N}$, resulting in a holomorphic $\mathrm{SU}(N)$ isometry of the metric $g$. For $k \geq 2$, there is a further isometry of $\tilde{\mathcal{V}}_{k,N}$ resulting from spatial rotations of the vortices.

Let us examine the moduli space of a single vortex in further detail. Given a specific solution $(B_\star, q_\star)$ to the abelian vortex equations, one can always construct a solution to the non-abelian equations (2.3) by simply embedding $(B_\star, q_\star)$ in the upper-left corner of the $N \times N$ matrices $B$ and $q$. In the case of a single vortex $k = 1$, acting on this configuration with the $\mathrm{SU}(N)_{\mathrm{diag}}$ symmetry sweeps out the full moduli space of solutions. Since the vortex embedded in the upper-left corner breaks $\mathrm{SU}(N)_{\mathrm{diag}} \to \mathrm{SU}(N-1) \times \mathrm{U}(1)$, the vortex moduli space is

$$\tilde{\mathcal{V}}_{1,N} \cong \mathrm{SU}(N)/(\mathrm{SU}(N-1)\times \mathrm{U}(1)) \cong \mathbf{CP}^{N-1}$$

endowed with the round Fubini-Study metric. The only information that we still need to determine is the overall scale of the moduli space. This will be important later in matching to the instanton moduli space. Since $\mathbf{CP}^{N-1}$ is a homogeneous space, we can fix the scale by calculating the overlap of *any* two suitable zero modes arising from the $\mathrm{SU}(N)_{\mathrm{diag}}$ action. For $\Omega(z,\bar{z}) \in su(N)$, the zero modes associated to an $\mathrm{SU}(N)_{\mathrm{diag}}$ rotation are given by,

$$\dot{A} = \mathcal{D}^a\Omega\,, \qquad \dot{q} = i(\Omega q - q\Omega_0)\,, \tag{2.9}$$

where $\Omega \to \Omega_0$ as $|z| \to \infty$. The transformation of $q$ arises because the left action is by the $\mathrm{U}(N)_G$ gauge symmetry, while the right action is by the $\mathrm{SU}(N)_F$ flavour symmetry. The $z$ dependence of $\Omega$ is required in order to satisfy the gauge fixing condition (2.6) which becomes

$$(\mathcal{D}^a)^2\Omega = e^2(\{\Omega, qq^\dagger\} - 2q\,\Omega_0\,q^\dagger)\,.$$

For the initial configuration embedded in the upper-left corner of $B$ and $q$, these equations are solved by the $(N-1)$ rotations,

$$(\Omega^a{}_b)^j = \left(\frac{q_\star}{\sqrt{\zeta}}\right)\delta^a{}_1\delta^j{}_b + \left(\frac{q_\star^\dagger}{\sqrt{\zeta}}\right)\delta^{ja}\delta_{b1} \qquad j = 2,\ldots,N$$

and it is a simple matter to compute the overlap (2.8) of the zero modes (2.9) to determine the overall radius of the moduli space to be

$$\mathrm{Radius}^2\left(\tilde{\mathcal{V}}_{1,N}\right) \sim \frac{1}{e^2}\,. \tag{2.10}$$

Finally, let us make a brief comment on the spectrum of vortices in the quantum theory. In the $d = 2+1$ theory with $\mathcal{N} = 4$ supersymmetry, ground states of the vortices in a given sector are associated to harmonic forms on $\tilde{\mathcal{V}}_{k,N}$. For the case of a single vortex, there are therefore $\chi(\mathbf{CP}^{N-1}) = N$ such states, implying that the vortex transforms in the fundamental representation of $\mathrm{SU}(N)_{\mathrm{diag}}$.

## 3. Branes

In this section, we discuss a brane realisation of the vortices in type IIB string theory. We start with the $d = 2+1$, $\mathcal{N} = 4$ $\mathrm{U}(N)$ Yang-Mills-Higgs theory described in the lagrangian (2.1). The brane realisation of this is well known [16] and consists of $N$ D3 branes, suspended between two parallel NS5-branes. A further $N$ semi-infinite $D3$ branes connect to the right-hand NS5-brane to provide the hypermultiplets.

In figure 1 we draw this brane configuration, firstly on the Coulomb branch with $\zeta = 0$, and secondly on the Higgs branch in which one NS5-brane is separated from the other branes, inducing a non-zero FI parameter $\zeta$. In the second picture, we also include the BPS vortices which appear as $k$ D1-branes stretched between the D3-branes and the isolated NS5-brane. To see that these D1-branes are indeed identified with vortices, note that they are the only BPS states of the brane configuration with the correct mass. The spatial worldvolume directions of the branes follow official convention:

$$\begin{aligned} NS5:&\ \ 12345 \\ D3:&\ \ 126 \\ D1:&\ \ 9 \end{aligned}$$

Both the FI parameter $\zeta$, and the gauge coupling, are encoded in the separation $\Delta x$ between the two NS5-branes. We have

$$\frac{1}{e^2} = \frac{\Delta x^6}{2\pi g_s}\,; \qquad \zeta = \frac{\Delta x^9}{4\pi^2 g_s l_s^2}\,, \tag{3.1}$$

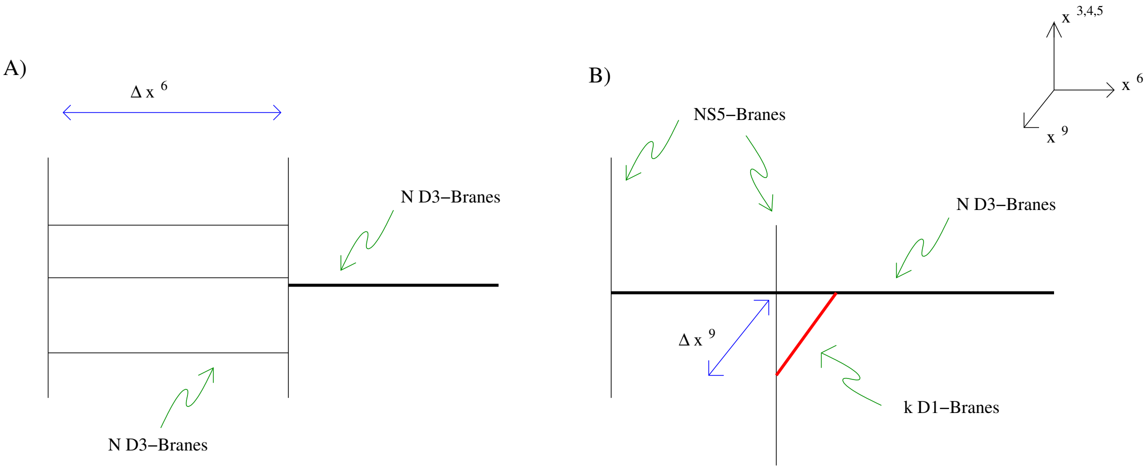


**Figure 1:** The brane configuration for U($N$) gauge theory with $N$ hypermultiplets. Figure 1A shows the theory on the Coulomb branch. In Figure 1B, the theory has a FI parameter and lies in its unique ground state. The D1-branes are the vortices.

where $l_s = \sqrt{\alpha'}$ and $g_s$ are the string length and coupling respectively. To take the gauge theory decoupling limit, we want to send $g_s \to 0$, while insisting that the field theory excitations are much smaller than other stringy and Kaluza-Klein modes. The two mass scales of the field theory are the mass of the photon $M_\gamma \sim \sqrt{e^2\zeta}$ and the mass of the vortex $M_v \sim \zeta$. An interesting curiosity about vortices is that while their mass is $M_v$, their size is $M_\gamma^{-1}$. In order to decouple the gauge theory from the string dynamics, we require

$$M_\gamma\,,\, M_v \ll \frac{1}{l_s}\,,\, \frac{1}{\Delta x^6} \tag{3.2}$$

while the ratio $(M_v/M_\gamma)^2 \sim \Delta x^6 \Delta x^9 / l_s^2 g_s^2$ remains fixed. The decoupling limit can therefore be achieved by setting $\Delta x^6 \sim \epsilon l_s$ and $\Delta x^9 \sim \epsilon^3 l_s$ and $g_s \sim \epsilon^2$, taking $\epsilon \to 0$.

Let us now turn to the vortices. It is a simple matter to read off the theory living on the worldvolume of the D1-branes (similar configurations were considered previously in the T-dual picture [17, 18]). The dynamics of the D1-branes is controlled by an $\mathcal{N} = (2,2)$ supersymmetric, gauged quantum mechanics. The relevant representations of the supersymmetry algebra are simply the dimensional reduction of the familiar vector and chiral multiplets in $d = 3+1$ dimensions. The vortex theory involves a U($k$) vector multiplet, consisting of a gauge field together with three adjoint scalar fields $\phi^r$, $r = 1,2,3$ parameterising the motion of the D1-branes in the $x^{r+2}$ directions. These are coupled to an adjoint chiral multiplet whose complex scalar we denote $Z$. The eigenvalues of $Z$ parameterise the position of the $k$ D1-branes in the $z = x^1 + ix^2$ plane. A further $N$ fundamental chiral multiplets, with complex scalars $\psi$, arise from the D1-D3 strings. The global symmetry group of the theory is

$$G = \mathrm{SU}(2)_R \times \mathrm{SU}(N)_D \times \mathrm{U}(1)_F\,, \tag{3.3}$$

where $\mathrm{SU}(2)_R$ is an R-symmetry rotating the scalars in the vector multiplet,[5] $\mathrm{U}(1)_F$ is

[5]For vortex solutions whose worldvolume is $d$-dimensional, this R-symmetry group is Spin$(4-d)$.



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a flavour symmetry rotating the phase of $Z$ and $SU(N)_D$ is a flavour symmetry acting on $\psi$ in the anti-fundamental representation. The $\psi$ fields may be represented as $k \times N$ matrices, with the $U(k)$ gauge group acting by left multiplication, and the $SU(N)_D$ flavour symmetry acting by right multiplication. We use the notation

$$\psi = \psi^m_{\ i} \qquad m = 1, \ldots, k \ ; \ i = 1, \ldots, N\,.$$

All of these fields come with fermionic superpartners which we suppress. The bosonic lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\rm vort} = \mathrm{Tr}\Bigg[&\frac{1}{2g^2}\mathcal{D}_t\phi^r\mathcal{D}_t\phi^r + \mathcal{D}_tZ^\dagger\mathcal{D}_tZ + \mathcal{D}_t\psi_i\mathcal{D}_t\psi^{i\dagger} - \frac{1}{2g^2}[\phi^r,\phi^s]^2 - \\ &- |[Z,\phi^r]|^2 - \psi\psi^\dagger\phi^r\phi^r - \frac{g^2}{2}\left(\psi\psi^\dagger - [Z,Z^\dagger] - r\mathbf{1}_k\right)^2\Bigg]\,. \end{aligned} \tag{3.4}$$

Once again, the gauge coupling $g^2$ and FI parameter $r$ of this theory are determined by the separation of the NS5-branes, although with reciprocal relations to the D3-brane theory (3.1)

$$\frac{1}{g^2} = \frac{2\pi l_s^2\,\Delta x^9}{g_s}\,; \qquad r = \frac{\Delta x^6}{g_s}\,. \tag{3.5}$$

We see that taking the decoupling limit of the D3-brane theory implies the strong coupling limit of the vortex theory $g^2 \to \infty$. However, the FI parameter $r$ remains finite and in fact is identified with the gauge coupling $e^2$

$$r = \frac{2\pi}{e^2}\,. \tag{3.6}$$

For $r \neq 0$, there is no Coulomb branch, so that taking the strong coupling limit $g^2 \to \infty$ decouples the vector multiplet fields $\phi^r$ and restricts attention to the Higgs branch of the theory. We shall denote this Higgs branch as $\mathcal{M}_{k,N}$. It is given by a $U(k)$ Kähler quotient of $\mathbf{C}^{k(N+k)}$, parameterised by $Z$ and $\psi$. The associated moment map is simply the D-term from (3.4)

$$D^m_{\ n} = \psi^m_{\ i}\psi^{i\dagger}_{\ n} - [Z,Z^\dagger]^m_{\ n} - r\delta^m_{\ n} = 0\,. \tag{3.7}$$

This imposes $k^2$ real constraints on $\mathbf{C}^{k(N+k)}$, while modding out by the $U(k)$ gauge group reduces the dimension of the Higgs branch by another $k^2$. Thus the real dimension of the Higgs branch is

$$\dim(\mathcal{M}_{k,N}) = 2kN$$

which we recognise as the dimension of the vortex moduli space (2.4). Indeed, the main result of this paper is the brane-predicted isomorphism

$$\mathcal{V}_{k,N} \;\cong\; \mathcal{M}_{k,N}\,. \tag{3.8}$$

### 3.1 Some examples and the metric

Let us examine the claim (3.8) in more detail. Firstly, note that the center of mass position of the D1-branes, given by $Z = z\mathbf{1}$, decouples from the other fields, guaranteeing that the Higgs branch decomposes as

$$\mathcal{M}_{k,N} \;\cong\; \mathbf{C} \times \tilde{\mathcal{M}}_{k,N} \tag{3.9}$$

in agreement with the vortex moduli space. To make further comparisons, let us consider specific examples, starting with the description of a single vortex $k = 1$ in the $\mathrm{U}(N)$ theory. In this case the vortex dynamics is abelian so $Z$ decouples and the D-term constraint reduces to $|q|^2 = r$ where $q$ is an $N$-vector. We are left with the well known gauged linear sigma-model construction of $\mathbf{CP}^{N-1}$, and we have,

$$\tilde{\mathcal{M}}_{1,N} \;\cong\; \mathbf{CP}^{N-1} \;\cong\; \tilde{\mathcal{V}}_{1,N}\,. \tag{3.10}$$

The size, or Kähler class, of the Higgs branch is determined by the FI parameter $r = 2\pi/e^2$ in agreement with the vortex moduli space (2.10).

The second example we consider is that of $k$ vortices in the abelian-Higgs model with $N = 1$. This vortex quantum mechanics was previously studied in [19] as a matrix model for identical particles moving on the plane. Prior to that, the D-term constraints (3.7) were solved in a somewhat different context by Polychronakos [20], who showed that a given solution is uniquely determined by a set of eigenvalues for $Z$, up to Weyl permutations. Thus

$$\mathcal{M}_{k,1} \;\cong\; \mathbf{C}^k/S_k \;\cong\; \mathcal{V}_{k,1}\,. \tag{3.11}$$

In these two, simple cases, we have therefore confirmed that the Higgs branch and vortex moduli spaces are indeed isomorphic. We now turn to the question of the metric. The Higgs branch $\mathcal{M}_{k,N}$ inherits a natural Kähler metric from the Kähler quotient construction described above. The presence of the flavour symmetry $\mathrm{SU}(N)_D$ guarantees that this metric exhibits an $\mathrm{SU}(N)$ holomorphic isometry. For $k \geq 2$, $\tilde{\mathcal{M}}_{k,N}$ also enjoys a $\mathrm{U}(1)$ holomorphic isometry, arising from $\mathrm{U}(1)_F$, corresponding to rotating the branes in the $x^1 + ix^2$ plane. Thus the quotient metric on $\mathcal{M}_{k,N}$ and the Manton metric on $\mathcal{V}_{k,N}$ share the same isometries. Indeed, from the brane picture it is clear that the $\mathrm{SU}(N)_{\mathrm{diag}}$ and $\mathrm{SU}(N)_D$ symmetry groups of the D3-brane and D1-brane theories, share the same origin.

Do further properties of the metrics coincide? In the case of $k = 1$, the metric on $\tilde{\mathcal{M}}_{1,N}$ is the round Fubini-Study metric on $\mathbf{CP}^{N-1}$, in agreement with the Manton metric on $\tilde{\mathcal{V}}_{1,N}$. However, in this case the agreement is a consequence of the symmetries of the problem. In general, the metrics are not the same. To see this, let us return to the case of the abelian-Higgs model with $N = 1$. Importantly, the asymptotic metric on $\mathcal{M}_{k,1}$ is the flat metric on $\mathbf{C}^k/S_k$, in agreement with the Manton metric. This is crucial to ensure that the Higgs branch describes the moduli space of indistinguishable particles since mere topological information does not suffice (topologically $\mathbf{C}^k/S_k \cong \mathbf{C}^k$ as any polynomial will confirm). However, in the case of the Kähler quotient, the leading order corrections to the flat metric are power-law. This is to be contrasted with the exponential corrections of the Manton metric. To be concrete, consider the case $k = 2$, $N = 1$. The metrics on both $\tilde{\mathcal{V}}_{2,1}$

and $\tilde{\mathcal{M}}_{2,1}$ take the form,

$$ds^2 = f^2(\sigma)(d\sigma^2 + \sigma^2 d\theta^2)\,, \tag{3.12}$$

where $\sigma$ is the separation between vortices, or D1-branes, and $\theta \in [0, \pi)$ so that the moduli space looks like a cone. For the Higgs branch, the explicit Kähler quotient construction was performed in [19] and the conformal factor is given by,

$$f^2_{\mathcal{M}}(\sigma) = \frac{\sigma^2}{\sqrt{\sigma^4 + r^2}} \approx 1 - \frac{r^2}{2\sigma^4} + \cdots\,. \tag{3.13}$$

The calculation of the leading order scattering of vortices was performed in [6], and the equivalent metric on $\tilde{\mathcal{V}}_{2,1}$ was computed to be,

$$f^2_{\mathcal{V}}(\sigma) \approx 1 - \lambda^2 \sqrt{\frac{4\pi}{\sigma}} e^{-2\sigma} + \cdots\,, \tag{3.14}$$

where $\lambda$ is a coefficient which parameterises the asymptotic return to vacuum of the Higgs field in the solution to (2.3). This coefficient is not known analytically but it was shown in [14] that T-duality between the $A_N$ singularity and *fully localised* NS5-branes requires a worldsheet instanton effect and holds only if $\lambda = 8^{1/4} \approx 1.682$. This is in agreement with the numerical result $\lambda \approx 1.683$ of [15]. To summarise, we see that, while the metrics on $\mathcal{V}_{k,1}$ and $\mathcal{M}_{k,1}$ are asymptotically, and qualitatively, similar they differ in the details.

## 4. Instantons

The Kähler quotient construction of the vortex moduli space is reminiscent of the hyperKähler quotient of the moduli space of $k$ instantons in U($N$) Yang-Mills theory. We denote this latter space as $\mathcal{I}_{k,N}$. In this section, we make the connection between $\mathcal{V}_{k,N}$ and $\mathcal{I}_{k,N}$ more explicit. We begin with a review of the ADHM gauge theory describing instantons on non-commutative $\mathbb{R}^4$, with the specific anti-self-dual, commutation relations

$$[x^1, x^2] = i\theta\,, \qquad [x^3, x^4] = -i\theta \tag{4.1}$$

with all other commutators vanishing. Recall that the ADHM construction of $\mathcal{I}_{k,N}$, as proposed in [2], can be elegantly described in terms of an auxiliary U($k$) gauge theory with 8 supercharges [21]. The matter content of this theory includes an adjoint valued hypermultiplet and $N$ fundamental hypermultiplets. The instanton moduli space is described as a hyperKähler quotient as the Higgs branch of this gauge theory, parameterised by the hypermultiplet scalar fields. Denote the two complex scalars in the adjoint multiplet as $Z$ and $W$, and the $2N$ complex scalars in the remaining hypermultiplets as $\psi$ and $\tilde{\psi}$. While $\psi$ transforms in the $\mathbf{k}$ representation of the gauge group, $\tilde{\psi}$ transforms as $\bar{\mathbf{k}}$, and we represent both of these fields as a $k \times N$ (respectively $N \times k$) matrix,

$$\psi = \psi^m_{\ i}\,, \qquad \tilde{\psi} = \tilde{\psi}^i_{\ m}\,, \qquad m = 1, \ldots, k,\ i = 1, \ldots, N\,. \tag{4.2}$$

Theories with 8 supercharges have a triplet of D-terms which, in 4 supercharge language, can be decomposed into a D-term and F-term. These constraints, which provide the triplet of moment maps in the hyperKähler quotient construction, read

$$\begin{aligned} D^m_{\ n} &= \psi^m_{\ i}\psi^{i\dagger}_{\ \ n} - \tilde{\psi}^{m\dagger}_{\ \ i}\tilde{\psi}^i_{\ n} - [Z,Z^\dagger]^m_{\ n} - [W,W^\dagger]^m_{\ n} - r\delta^m_{\ n} = 0 \\ F^m_{\ n} &= \psi^m_{\ i}\tilde{\psi}^i_{\ n} + i[Z,W]^m_{\ n} = 0\,. \end{aligned}$$

The FI parameter $r$ appears only in the D-term, a fact related to the specific choice of non-commutative background (4.1) as shown by Nekrasov and Schwarz [22]. The relationship is simply

$$r = 4\theta\,.$$

The role of $r$ is to resolve the singularities of $\mathcal{I}_{k,N}$ in the manner proscribed by Nakajima [23]. In doing so, it picks out a preferred complex structure on $\mathcal{I}_{k,N}$.

Note that we have used the same notation in the ADHM gauge theory as we did in the the previous section, and we will shortly explain the deformation which takes us from ADHM to the vortex theory. Before doing so, it will do us well to dwell a little on the symmetries of the ADHM theory. To compare with the previous section we choose to define the ADHM theory in $d = 0+1$ dimensions, describing particles in $d = 4+1$ dimensional Yang-Mills or, alternatively, D0-branes moving in the background of D4-branes. The global symmetry group of the ADHM theory is

$$G' = \mathrm{Spin}(5)_R \times \mathrm{SU}(N)_F \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_R\,. \tag{4.3}$$

The $\mathrm{Spin}(5)_R$ symmetry rotates the scalars in the vector multiplet.[6] The $\mathrm{U}(1)_R \times \mathrm{SU}(2)_L$ is what remains of the $SO(4)$ spatial rotation group of $\mathbb{R}^4$ with the anti-self-dual non-commutative deformation (4.1), and the $\mathrm{SU}(N)_F$ descends from the $\mathrm{U}(N)$ gauge symmetry on the D4-branes. The adjoint doublet $(Z,W)$ transforms as $(\mathbf{1},\mathbf{1},\mathbf{2})_{+1}$ under $G'$, while $\psi$ transforms as $(\mathbf{1},\bar{\mathbf{N}},\mathbf{1})_{+1}$ and $\tilde{\psi}$ transforms as $(\mathbf{1},\mathbf{N},\mathbf{1})_{+1}$, where the subscripts denote the charge $Q_R$ under the $\mathrm{U}(1)_R$ R-symmetry.

We are now in a position to describe the deformation which takes us to the vortex theory by adding masses to all the unnecessary fields. We accomplish this by *weakly gauging* a particular $\widehat{\mathrm{U}(1)}$ symmetry. This involves gauging a symmetry in a manner consistent with supersymmetry. The scalars in this new vector multiplet are then endowed with vacuum expectations values (vevs) and the new vector multiplet is subsequently decoupled. The only remnant of the whole process is the vevs, which give masses to any field charged under the $\widehat{\mathrm{U}(1)}$ symmetry. If $\widehat{\mathrm{U}(1)}$ is taken to be a flavour symmetry, then this process preserves the full 8 supercharges of the ADHM theory. In contrast, if $\widehat{\mathrm{U}(1)}$ is a generic R-symmetry, this process breaks all supersymmetry. However, there are specific combinations of R-symmetries which one may gauge which preserve a fraction of the supersymmetry and it is this combination that we shall employ. Let $\mathrm{U}(1)_A \subset \mathrm{Spin}(5)$ be such that it rotates two of the vector multiplet scalars, leaving the remaining three untouched; let $\mathrm{U}(1)_L \subset \mathrm{SU}(2)_L$ have the Pauli matrix generator $\tau^3$; and let $\mathrm{U}(1)_G \subset \mathrm{U}(k)$ be the overall gauge rotation. Then we choose the combination of symmetries that act on fields with charge $Q$, such that

$$\widehat{Q} = Q_A + Q_R - Q_L - Q_G\,. \tag{4.4}$$

[6]For instantons with a $d$-dimensional worldvolume, this R-symmetry group is $\mathrm{Spin}(6-d)$.

The fields $W$ and $\tilde{\psi}$, together with two of the five vector multiplet scalars, have $\widehat{Q} \neq 0$. These all receive masses. The fields $Z$ and $\psi$, and the three remaining scalars of the vector multiplet all have $\widehat{Q} = 0$ and survive unscathed. We are left with the vortex theory of section 2, with the relationship between the FI parameter and parameters giving,

$$\theta = \frac{\pi}{2e^2} .$$

### 4.1 The moduli spaces

While the above discussion has been in terms of the ADHM gauge theory, the deformation also has a simple description directly in terms of the instanton moduli space $\mathcal{I}_{k,N}$. The $\widehat{\mathrm{U}(1)}$ symmetry of the gauge theory descends to an $\hat{\mathbf{S}}^1$ action on $\mathcal{I}_{k,N}$, endowing the metric on $\mathcal{I}_{k,N}$ with a Killing vector $\hat{k}$. This Killing vector is holomorphic, preserving the preferred complex structure while rotating the remaining two. The mass terms introduced above by *weakly gauging* $\widehat{\mathrm{U}(1)}$ induce to a potential $V$ on $\mathcal{I}_{k,N}$ proportional to the length$^2$ of the Killing vector,

$$V \sim \hat{k}^2 . \qquad (4.5)$$

Such potentials have been widely used in soliton physics recently (see for example [24],[25]), although usually in the context of supersymmetry-preserving tri-holomorphic Killing vectors. We therefore have a description of the vortex moduli space $\mathcal{V}_{k,N}$ directly in terms of the instanton moduli space

$$\mathcal{V}_{k,N} \;\cong\; \mathcal{I}_{k,N} \Big|_{\hat{k}=0} .$$

The zeroes of the Killing vector $\hat{k}$ are precisely the fixed points of the $\hat{\mathbf{S}}^1$ action. The meaning of this action can be determined from the assignment of charges $\hat{Q}$ in (4.4). Recall that $\mathrm{U}(1)_R \times \mathrm{U}(1)_L \subset \mathrm{SU}(2)_R \times \mathrm{U}(1)_L$ is the subgroup of the rotations $SO(4) \cong \mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ of $\mathbb{R}^4$ that are left unbroken by the non-commutative deformation (4.1). We find therefore that the action $\hat{Q}$ corresponds to rotating the instantons in the $x^3 - x^4$ plane, and the vortices are related to instantons which are invariant under this $\mathrm{U}(1)$ action.

Let us now turn to some examples: the moduli space $\tilde{\mathcal{I}}_{1,N}$ of a single instanton in $\mathrm{U}(N)$ non-commutative Yang-Mills is given by the cotangent bundle $T^\star(\mathbf{CP}^{N-1})$ endowed with the Calabi metric [26]. The potential $\hat{k}^2$ vanishes on the zero section of the bundle $\mathbf{CP}^{N-1}$, reducing to the moduli space of a single vortex $\tilde{\mathcal{V}}_{1,N}$. Another example: the moduli space $\tilde{\mathcal{I}}_{2,1}$ of two instantons in $\mathrm{U}(1)$ gauge theory is the Eguchi-Hanson metric on $T^\star(\mathbf{S}^2)$. The explicit hyperKähler quotient construction was performed in [27]. Note that this case is special since $\tilde{\mathcal{I}}_{2,1} \cong \tilde{\mathcal{I}}_{1,2}$, which is not true for $k > 2$. However, the tri-holomorphic $\mathrm{SU}(2)$ isometry of $T^\star(\mathbf{S}^2)$ has a different origin in these two cases. In the notation of (4.3), the isometry is $\mathrm{SU}(2)_F$ for $\tilde{\mathcal{I}}_{1,2}$, while it is $\mathrm{SU}(2)_L$ for $\tilde{\mathcal{I}}_{2,1}$. Since, from (4.4), the potential on the instanton moduli space involves $\mathrm{SU}(2)_L$, but not $\mathrm{SU}(2)_F$, the vortex moduli spaces $\tilde{\mathcal{V}}_{2,1}$ and $\tilde{\mathcal{V}}_{1,2}$ are given by different holomorphic submanifolds of $T^\star(\mathbf{S}^2)$. It is a simple exercise to show that the vacua of the potential on $\tilde{\mathcal{I}}_{2,1}$ is the two dimensional cone endowed with the metric (3.13).



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### 4.2 A wrapped brane realisation

From the perspective of the D4-brane, the above deformation of the instanton theory involves locking the $U(1)_{L/R}$ symmetries tangent to the D4-brane, with the $U(1)_A$ symmetry normal to the D4-branes. This is reminiscent of the twisting of the tangent and normal bundles of branes when wrapped on cycles [28]. In this section, we give evidence suggesting that the two are indeed related.

To see this connection, let us first return to the brane set-up of section 3 as depicted in Figure 1. We perform a T-duality in the $x^9$ direction, and describe the resulting IIA string theory set-up. Under T-duality, the two NS5-branes are replaced by the background geometry $\mathbf{C}^2/\mathbf{Z}_2$. (The duality between NS5-branes and *ALF* spaces was first conjectured by Hull and Townsend [29]. A proof from the worldsheet sigma-model, including the breaking of translation symmetry associated to the localization of the NS5-brane, was given in [14]). The separation of the NS5-branes in the $x^6$ direction resolves the orbifold singularity, resulting in the background spacetime $T^\star(\mathbf{S}^2)$.[7] Topologically, this space can be thought of as an $\mathbf{S}^1$ fibration, parameterised by $x^9$, over $\mathbb{R}^3$, parameterised by $\mathbf{r} = (x^6, x^7, x^8)$. In Gibbons-Hawking coordinates, the metric takes the form,

$$ds^2 = H(r)\,dr^2 + \tfrac{1}{4}H(r)^{-1}(dx^9 + \boldsymbol{\omega}\cdot d\mathbf{r})^2\,,$$

where $\nabla\times\boldsymbol{\omega} = \nabla H$ and

$$H(r) = \frac{1}{|\mathbf{r}-\mathbf{r}_0|} + \frac{1}{|\mathbf{r}+\mathbf{r}_0|}\,.$$

The 3-vector $\mathbf{r}_0$ resolves the orbifold singularity and, for the T-dual of Figure 1, is given by $\mathbf{r}_0 \sim (1/e^2, 0, 0)$. The $\mathbf{S}^1$ fiber degenerates at the two points $\mathbf{r} = \pm\mathbf{r}_0$, resulting in the Christmas cracker topology shown in figure 2. The zero section $\mathbf{S}^2$, which can be clearly seen in this picture, contains a paper hat and a 20 year old joke.

What becomes of the D-branes after T-duality? The D3-branes of Figure 1 become D4-branes with worldvolume spanning $x^1, x^2, x^6$ and $x^9$. They wrap the compact $S^2$, and one half of the cracker as depicted by shading in the Figure. The vortices are a little more mysterious. Had the D1-branes in Figure 1 been infinite in the $x^9$ direction, they would become D0-branes in the IIA description. Since the D1-branes actually stretch only a fraction of the distance, we expect that they become fractional D0-branes. However, such objects are usually understood in terms of a D2-$\bar{\text{D}}$2 pair wrapping a vanishing $\mathbf{S}^2$, through which an NS-NS B-field threads in order to provide the D0-brane charge. Yet in our case the $\mathbf{S}^2$ has finite size, and such an interpretation breaks down, as can easily be seen by computing the mass of such a D2-$\bar{\text{D}}$2 pair. It would be interesting to get a better understanding of these fractional D0-branes in this picture, and complete the relationship to the wrapped D0-D4 system.

[7] Note that this ubiquitous space has already appeared twice as the instanton moduli spaces $\tilde{\mathcal{I}}_{1,2}$ and $\tilde{\mathcal{I}}_{2,1}$. Here it appears in an unrelated context as the background spacetime in string theory.

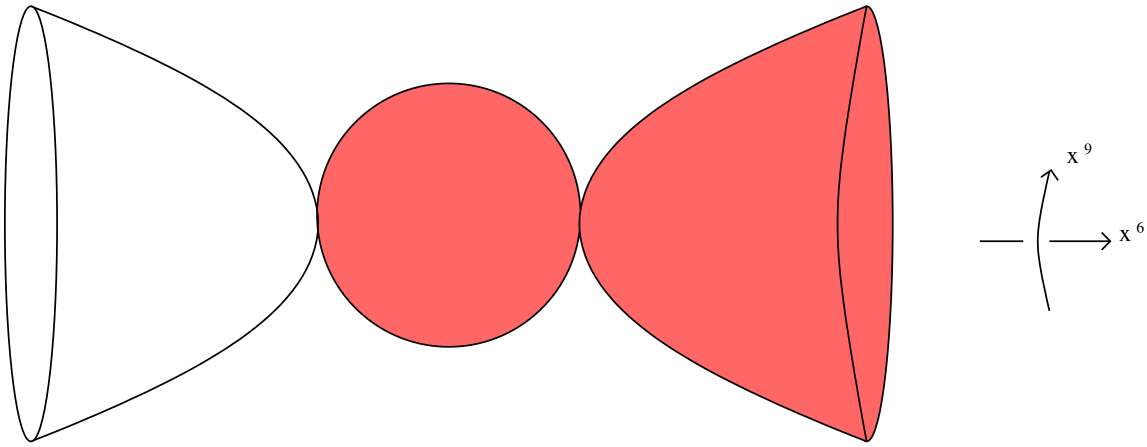


**Figure 2:** The Eguchi-Hanson Christmas cracker. The D4-branes are wrapped around the shaded region.



## 5. Semi-local vortices and sigma-model lumps

In this section, we discuss a generalisation of the vortices to U($N$) Yang-Mills with $N_f = (N+M)$ flavours. The lagrangian takes the same form as previously (2.1) except the fundamental scalars are now $N \times (N+M)$ dimensional matrices

$$q = q^a{}_i\,, \qquad \tilde{q} = \tilde{q}^i{}_a \qquad a = 1\ldots, N\,, \qquad i = 1, \ldots, N+M\,.$$

Rather than the unique, isolated vacuum of section 2, the theory now has a Higgs branch of vacua. However, if $\tilde{q}$ develops an expectation value, then there are no BPS vortex solutions. This may be easily seen from an analysis of the Bogomoln'yi equations, and follows from the mathematical fact truth that a line bundle of negative degree admits no holomorphic sections (see, for example, [30] for the translation). We therefore restrict attention to the reduced Higgs branch of vacua, denoted $\mathcal{N}_{N,M}$ obtained by insisting $\tilde{q} = 0$. For example, for abelian theories with $N = 1$, the Higgs branch of vacua is the cotangent bundle $T^\star(\mathbf{CP}^M)$, while the reduced Higgs branch describing the vacua which admit BPS vortex solutions is simply $\mathcal{N}_{1,M} = \mathbf{CP}^M$. In general, the reduced Higgs branch is the grassmannian of $N$ planes in $\mathbf{C}^{N+M}$,

$$\mathcal{N}_{N,M} = G(N, N+M)\,. \tag{5.1}$$

This is a symmetric space, and we may choose to work in any of the vacua without loss of generality. We pick,

$$\begin{aligned} q^a{}_i &= \sqrt{\zeta}\delta^a{}_i && i = 1, \ldots N \\ q^a{}_i &= 0 && i = N+1, \ldots N+M \\ \tilde{q}^i{}_a &= 0 && i = 1, \ldots, N+M\,. \end{aligned} \tag{5.2}$$

In this vacuum the SU$(N+M)_F$ flavour symmetry of the theory is broken in the pattern,

$$\mathrm{U}(N)_G \times \mathrm{SU}(N+M)_F \to S\left[\mathrm{U}(N)_{\mathrm{diag}} \times \mathrm{U}(M)_F\right]. \tag{5.3}$$

The theory admits BPS vortices with the Bogomoln'yi equations taking the same form as previously (2.3) with $q$ now interpreted as a matrix of the appropriate size. We denote the moduli space of vortices in this model as $\hat{\mathcal{V}}_{k,(N,M)}$. Note that, in the notation of the previous sections, we have $\mathcal{V}_{k,N} \cong \hat{\mathcal{V}}_{k,(N,0)}$. It is a simple matter to generalise the index theorem of section 2 to the present case. We omit the details, stating only the result $\mathcal{J}(M^2 \to \infty) = k(N+M)$. We therefore have

$$\dim(\hat{\mathcal{V}}_{k,(N,M)}) = 2k(N+M)\,. \tag{5.4}$$

Note that since we have taken the limit $M^2 \to \infty$, this computation ignores a surface term contribution which comes from fields dropping off as $1/r$ [12]. Indeed, as we shall review below, it is known that the counting (5.4) includes zero modes which are not $L^2$ normalisable and which one would not, therefore, expect to be included in $\mathcal{J} = \mathcal{J}(M^2 = 0)$. Nevertheless, these modes corresponds to collective coordinates of the semi-local vortex and we wish to keep them in our discussion, so the result (5.4) is the relevant one.

In the abelian case $N = 1$, the vortex equations with multiple Higgs fields have been well studied in the literature, where they go by the name of *semi-local vortices*. For a review of their properties and their relationship to electroweak strings, see [31]. The result $\dim(\hat{\mathcal{V}}_{k,(1,M)}) = 2k(1+M)$ was previously determined from a direct analysis of the Bogomoln'yi equations in [32] and subsequently from a brane picture in [33].

An interesting feature of the semi-local vortices is that they may remain non-singular in the limit $e^2 \to \infty$. This is in contrast to the vortices considered in section 2 whose size scales as $(e^2\zeta)^{-1/2}$ and thus become point-like objects in this limit. In fact, the semi-local vortices reduce to another familiar topological object as $e^2 \to \infty$: they become sigma-model lumps (a.k.a sigma model instantons, or textures) on the reduced target space $\mathcal{N}_{N,M}$. While the vortices are supported by $\Pi_1(\mathrm{U}(N))$, the lumps are supported by $\Pi_2(\mathcal{N}_{N,M})$. For example, the semi-local vortices of the abelian $N = 1$ model become lumps on $\mathbf{CP}^M$. A nice description of how the metamorphosis from vortex to lump occurs may be found in [34]. Thus, in the limit $e^2 \to \infty$, the moduli space of semi-local vortices $\hat{\mathcal{V}}_{k,(N,M)}$ becomes the moduli space of grassmannian $G(N, N+M)$ lumps.

It is well known that sigma-model lumps share several properties with Yang-Mills instantons. In particular, they may have arbitrary size and, in the $e^2 \to \infty$ limit, $k$ of the collective coordinates of (5.4) may be thought of as the scales of the $k$ lumps. Since the lumps may have any size, they can also shrink to a singular solution. Thus, just like the unresolved instanton moduli spaces, the moduli space of lumps contains singularities. These singularities are removed by introducing a gauge field with a finite coupling $e^2$ and returning to the full vortex equations (2.3). In this way, the inverse gauge coupling $1/e^2$ plays a role in the vortex dynamics reminiscent of the non-commutivity parameter $\theta$ in Yang-Mills instantons. This similarity was previously noted in [30, 35], and we shall make the analogy more explicit in the following section.

The Manton metric on $\hat{\mathcal{V}}_{k,(N,M)}$ may be once again defined in terms of the overlap of zero modes. The resulting metric is Kähler and inherits a $S(\mathrm{U}(N) \times \mathrm{U}(M))$ holomorphic isometry from the surviving symmetry group (5.3), together with a further $\mathrm{U}(1)$ isometry from the rotational symmetry. However, the Manton metric on $\hat{\mathcal{V}}_{k,(N,M)}$ suffers from a sickness since some of the zero modes are (logarithmically) non-normalisable. This well known problem for lumps in the $\mathbf{CP}^M$ sigma-model [36] is not ameliorated by a finite

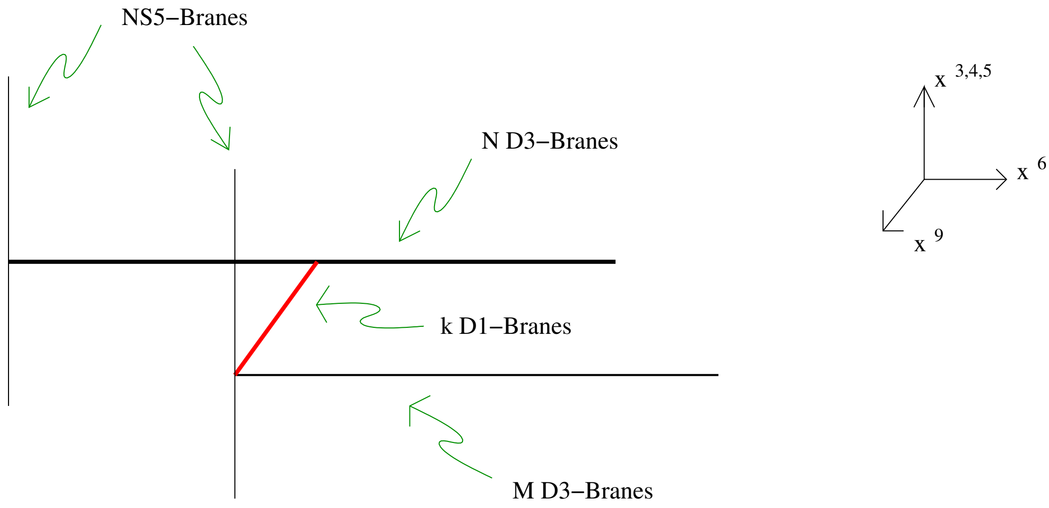


**Figure 3:** The brane configuration for U($N$) gauge theory with $N + M$ hypermultiplets, and $k$ vortices.

gauge coupling $e^2$ as shown in [37]. Classically this non-normalisability ensures that certain collective coordinates (for example, the scaling size of the lump) have infinite moment of inertia and are thus constants of the dynamics. The non-normalisability of modes leads to subtleties when treating these objects quantum mechanically.

### 5.1 Branes

We now turn to the brane realisation of these vortices. We keep the same basic structure as section 3, simply adding $M$ further semi-infinite D3-branes to provide the extra flavours. We choose to add these to the right-hand NS5-brane, so the final set-up is shown in figure 3.

Once again, it is a simple matter to read off the theory on the $k$ D1-branes [17]. It consists of a U($k$) field theory, still coupled to the chiral multiplets $Z$ and $\psi$ as in section 3, but now augmented with $M$ further chiral multiplets $\tilde{\psi}$ which transform in the $\bar{\mathbf{k}}$ representation of the gauge group. We shall write,

$$\tilde{\psi} = \tilde{\psi}^w_{\ m} \qquad m = 1, \ldots, k\,;\ w = 1, \ldots, M\,.$$

These fields also transform under their own $\mathrm{U}(M)_E$ flavour symmetry, so the full global symmetry group of the theory is therefore

$$G = \mathrm{SU}(2)_R \times S(\mathrm{U}(N)_D \times \mathrm{U}(M)_E) \times \mathrm{U}(1)_F\,,$$

where the overall U(1) of the $\mathrm{U}(N)_D \times \mathrm{U}(M)_E$ flavour symmetry lies in the U($k$) gauge group.

As in section 3, we are interested in the Higgs branch of the D1-brane theory, which we denote as $\hat{\mathcal{M}}_{k,(N,M)}$. This Higgs branch is expected to be isomorphic to the vortex moduli space,

$$\hat{\mathcal{V}}_{k,(N,M)} \cong \hat{\mathcal{M}}_{k,(N,M)}\,.$$

Let us examine the Higgs branch in more detail. It is given by a U($k$) quotient of $\mathbf{C}^{k(N+M+k)}$, parameterised by $Z$, $\psi$ and $\tilde{\psi}$. The D-term moment map is

$$D^m_{\ n} = \sum_{i=1}^{N} \psi^m_{\ i}\psi^{i\dagger}_{\ \ n} - \sum_{w=1}^{M} \tilde{\psi}^{m\dagger}_{\ \ w}\tilde{\psi}^w_{\ n} - [Z, Z^\dagger]^m_{\ n} - r\delta^m_{\ n} = 0\,, \tag{5.5}$$

where we have, for once, abandoned the summation convention in order to highlight the ranges of the various indices. The D-term imposes $k^2$ real constraints which are augmented by the restriction to U($k$) invariant coordinates. Thus the real dimension of the Higgs branch is

$$\dim(\hat{\mathcal{M}}_{k,(N,M)}) = 2k(N+M)$$

in agreement with the vortex moduli space. The symmetry group $G$ imprints itself as a holomorphic $S(\mathrm{U}(N)\times\mathrm{U}(M))\times\mathrm{U}(1)$ isometry of the Higgs branch and thus, as before, the symmetries of the metric on the Higgs branch defined by the Kähler quotient construction are the same as those of the Manton metric. However, as in section 3, here the agreement stops. In particular, the metric on $\hat{\mathcal{M}}_{k,(N,M)}$ defined by the Kähler quotient construction is finite, and sees nothing of the non-normalisable modes of the vortex. Given our remarks in the introduction, one would expect that for many supersymmetric problems in soliton and instanton physics, one can replace the Manton metric on $\hat{\mathcal{V}}_{k\,(N,M)}$ with the Kähler quotient metric on $\hat{\mathcal{M}}_{k,(N,M)}$ and in this manner avoid the subtleties of the non-normalisable directions.

The FI parameter of (5.5) is once again related to the gauge coupling,

$$r = \frac{2\pi}{e^2}$$

so that the limit in which the semi-local vortices reduce to sigma model lumps is simply $r \to 0$. Let examine how the Higgs branch changes with $r$. For the vortex theory described in section 3, the Higgs branch only exists for $r > 0$. When $r = 0$, the D-term (3.7) requires $\psi = 0$, and the metric on the Higgs branch becomes the flat, singular metric on $\mathbf{C}^k/S^k$. This reflects the fact that the vortices of section 2 become point-like objects when $e^2 \to \infty$. However, things are somewhat different for the semi-local vortices. In this case, the moduli space is smooth for $r > 0$, and again develops singularities when $r = 0$. These singularities correspond to the zero size sigma-model lumps. Yet, even for $r = 0$, there still exist solutions to the D-term equations (5.5) with $\psi\,,\,\tilde{\psi} \neq 0$. These correspond to the non-singular sigma-model lumps of finite size. Note that for semi-local vortices the Higgs branch defined by (5.5) even makes sense for $r < 0$. One may want to interpret this as a "continuation past infinite coupling" ($e^2 \to \infty$ and then some). In the following section we shall give a different interpretation for the regime $r < 0$.

### 5.2 Instantons

The moduli space $\hat{\mathcal{V}}_{k,(N,M)}$ of $k$ vortices in U($N$) theories with $N_f = (N+M)$ flavours is again a complex submanifold of the moduli space of instantons. This time we must look at $\mathcal{I}_{k,N_f}$ describing $k$ instantons in U($N_f$) Yang-Mills theory. The ADHM theory for these instantons was described in section 4 where, obviously, we must replace $N$ with $N_f$. So, for example, the global symmetry group of the ADHM theory is,

$$G' = \mathrm{Spin}(5)_R \times \mathrm{SU}(N_f)_F \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_R\,.$$

As in section 4, we define the submanifold describing vortices by weakly gauging a symmetry or, equivalently, by the fixed point set of a $\mathbf{S}^1$ action on $\mathcal{I}_{k,N_f}$. The new ingredient here is that the $\widehat{\mathrm{U}(1)}$ action includes a component from the $\mathrm{SU}(N_f)_F$ flavour symmetry. In fact, it will prove to be simpler to phrase the discussion in terms of $\mathrm{U}(1)_G \times \mathrm{SU}(N_f)_F = \mathrm{U}(N_f)$, where $\mathrm{U}(1)_G \subset \mathrm{U}(k)$ is the overall $\mathrm{U}(1)$ gauge symmetry. To this end, consider the Cartan subalgebra of $\mathrm{U}(N_f)$,

$$\prod_{i=1}^{N_f} \mathrm{U}(1)_F^{(i)}\,.$$

Write the associated charges as $Q_F^{(i)}$, where $i = 1, \ldots, N_f$. In this notation, the charge under the overall $\mathrm{U}(1)$ gauge symmetry is,

$$Q_G = \sum_{i=1}^{N_f} Q_F^{(i)}\,.$$

The theory describing semi-local vortices can be obtained from the ADHM theory by a weak gauging which gives mass to all fields carrying non-vanishing charge,

$$\widehat{Q} = Q_A + Q_R - Q_L - \sum_{i=1}^{N} Q_F^{(i)} + \sum_{i=N+1}^{M} Q_F^{(i)}$$

and the vortex moduli space $\hat{\mathcal{V}}_{k,(N,M)}$ is isomorphic to the fixed point set of the associated $\widehat{\mathrm{U}(1)}$ action on the instanton moduli space $\mathcal{I}_{k,N+M}$. In this case, the $\widehat{\mathrm{U}(1)}$ action arises from a simultaneous rotation of the instantons in the $x^3 - x^4$ plane, together with a gauge rotation in $\mathrm{U}(N_f)$.

## 6. Non-commutative vortices

In this section we examine our Yang-Mills-Higgs theories defined the spatial non-commutative plane

$$[x^1, x^2] = -i\vartheta \tag{6.1}$$

and ask how this deformation affects the moduli space of vortices. Various aspects of non-commutative vortices in the abelian Higgs model have been considered in [40]–[43]. Before recalling the results of these papers, let us start by regaling ourselves with the beautiful tale of non-commutative instantons. We have already covered this is section 4, but a good story is always worth retelling. It was shown by Nekrasov and Schwarz [22] that a non-commutative deformation of $\mathbb{R}^4$ as given, for example, in (4.1) induces a FI parameter $r = 4\theta$ in the ADHM instanton gauge theory. This FI term resolves the singularities of the instanton moduli space in the manner described previously by Nakajima [23]. One may expect that a similar phenomenon occurs in our vortex theory. However, we have seen that the job of resolving the singularities on the vortex moduli space is already adequately performed by the gauge coupling since the FI parameter is $r = 2\pi/e^2$. So what role could the non-commutative deformation (6.1) play? To avoid undue suspense, we shall first reveal the answer, followed by a derivation, and then an analysis of the consequences. We shall show that the effect of a non-commutative background is to change the FI parameter of the vortex theory to

$$r = 2\pi\left(\frac{1}{e^2} + \vartheta\zeta\right),$$

To see this, we return once more to the brane picture of section 3. We want now to deform the D3-brane dynamics so that at low energies it is described by the Yang-Mills-Higgs theory (2.1) defined on the non-commutative plane (6.1). The string theory background that achieves this feat is well known [38]: a background NS NS $B$ field is added, with components $B_{12} \neq 0$. We wish to understand the effect of this $B$ field on the dynamics of the D1-strings.

In fact, a very similar situation was analysed by Hashimoto and Hashimoto in [39]. These authors considered the situation of a D-string suspended between two D3-branes in a background $B$ field, a set-up which describes a monopole in non-commutative Yang-Mills. The basic physics is very simple to describe. The background $B$ field may be absorbed into the D3-brane as a magnetic flux $F_{12}$. The end of the D1-brane acts as a magnetic source in the D3-brane, and therefore experiences a force due to $F_{12}$. To see how force acts, recall that our D3-brane lies in the 0126 directions and note that $F_{12} = {}^\star F_{06}$. The magnetic end of the D-string therefore feels the same force as an electric charge in a background electric field $F_{06}$. In other words, the string end moves in the $x^6$ direction. This displacement continues until the force due to the $B$ field is canceled by the excess tension of the D1-brane. In our case, one end of the D1-brane is attached to the NS5-brane, and cannot move in the $x^6$ direction. The final configuration is therefore given by the tilted D-strings, as shown in figure 4. An analysis of the supersymmetry generators was performed in [39] which, translated to the present set-up, reveals that these tilted D1-branes continue to preserve four supercharges.

The effect of this tilt on the theory on the D1-branes is to change the FI parameter $r$, which is now given (up to a normalisation of $1/g_s$ — see (3.5)) by the distance between the end of the D1-brane and the left-hand NS5-brane. The displacement $\delta$ of the D1-brane from its original position was calculated in [39], and is given by

$$\delta = -\frac{\vartheta\,\Delta x^9}{2\pi l_s^2}\,.$$

With the parameters $e^2$ and $\zeta$ still defined in terms of the distances $\Delta x^6$ and $\Delta x^9$ between the NS5-branes as in (3.1), we find the promised result,

$$r = 2\pi\left(\frac{1}{e^2} + \vartheta\zeta\right). \tag{6.2}$$

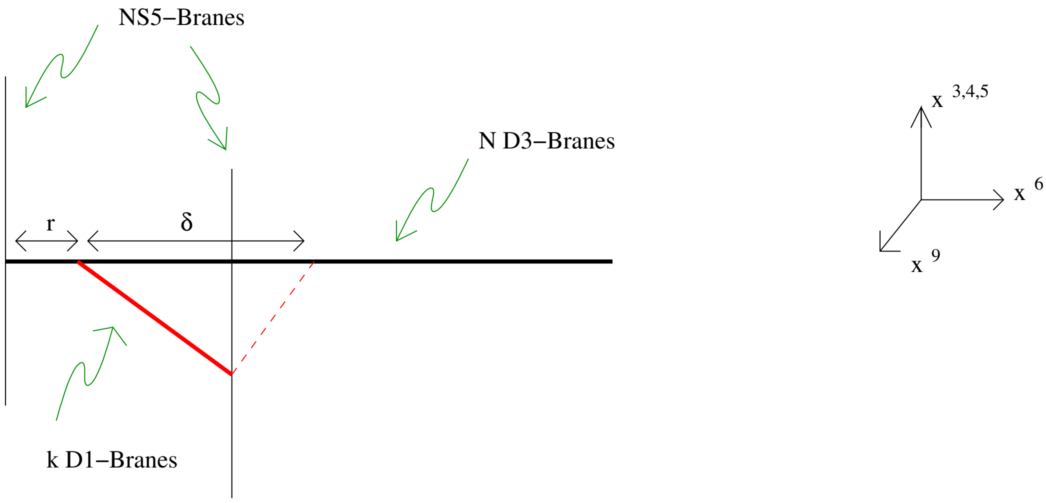


**Figure 4:** The brane configuration for vortices on a non-commutative background. A NS-NS B field lies in the $x^1 - x^2$ directions, inducing non-commutivity on the D3-brane worldvolume and causing the D1-brane to tilt.

We now turn to studying some simple consequences of this equation. Consider first the vortices of section 2, in which we have a U($N$) gauge group with $N$ hypermultiplets. The most striking feature is that the vortex moduli space only exists for $r \geq 0$ or, alternatively, for $\vartheta$ above the critical value $\vartheta_c$

$$\vartheta > \vartheta_c = -1/e^2\zeta$$

Moreover, at the critical value $\vartheta = \vartheta_c$, the moduli space becomes singular. For example, in the abelian case $N = 1$, we have $\mathcal{V}_{k,1} \cong \mathbf{C}^k/S_k$, endowed with the flat, singular metric at $\vartheta = \vartheta_c$. In fact neither of these results are new: both were previously derived in the abelian $N = 1$ theories by Bak, Lee and Park by an explicit study of the solutions to the non-commutative Bogomoln'yi equations [41] (see also the phase diagram in [43]). It is pleasing to see these properties reproduced from our D1-brane theory. From equation (6.2) it is also clear that vortex moduli space $\mathcal{V}_{k,N}$ is non-singular in the $e^2 \to \infty$ limit provided $\vartheta > 0$. This point was previously made in [43] and the Manton metric on $\mathcal{V}_{k,1}$ was explicitly calculated in this limit. Once again, the Manton metric differs from the metric induced on the Higgs branch by the Kähler quotient construction. For example, in the case of two vortices, the non-commutative metric in the $e^2 \to \infty$ limit takes the form of a cone (3.12), now with the conformal factor given by

$$f^2(\sigma) = \coth(\sigma^2/2) - \frac{\sigma^2}{2\sinh^2(\sigma^2/2)} \approx 1 - 2\sigma^2 e^{-\sigma^2} + \cdots$$

which coincides with neither (3.13) nor (3.14).

We now turn to the consequences of (6.2) for semi-local vortices. Firstly, note that if we set $e^2 \to \infty$ so that we are studying the moduli space of sigma-model lumps, then the non-commutivity parameter $\vartheta$ resolves the singularities just as in the case of Yang-Mills instantons. Similar observations were made from a field theory perspective in [44].

However, for semi-local vortices the moduli space defined by the D-term (5.5) continues to make sense for $r < 0$ or, alternatively, for $\vartheta < \vartheta_c$. Moreover, the topology of the moduli space differs for $r > 0$ and $r < 0$. Thus, as we decrease the non-commutivity parameter $\vartheta$ past its critical value of $-1/e^2\zeta$, the moduli space of non-commutative vortices undergoes a topology changing transition. A familiar example of this occurs for a single $k = 1$ semi-local vortex in a U(2) gauge theory with 4 hypermultiplets. In this case the moduli space is related to the Calabi-Yau 3-fold known to string theorists simply as The Conifold. For $r = 0$, it is the singular space defined by the complex equation $xy - wz = 0$. The FI parameter resolves the singularity and we have,

$$\hat{\mathcal{V}}_{1,(2,2)} \cong O(-1) \times O(-1) \to \mathbf{CP}^1 \, .$$

As we decrease the non-commutivity parameter $\vartheta$ to pass from $r > 0$ to $r < 0$, the moduli space of vortices undergoes a flop transition. For other values of $N$ and $M$, the topology change is more dramatic and the moduli space of vortices may have different Betti numbers for $r > 0$ and $r < 0$.

Finally, note that for $\vartheta < \vartheta_c$ the equation (6.2) implies a duality between different non-commutative gauge theories in the sense that they share the same moduli spaces of vortices. Specifically, consider the U($N$) theory with $N_f = (N + M)$ flavours. The moduli space of $k$ vortices $\hat{\mathcal{V}}_{k,(N,M)}$ is defined by the moment map (5.5) with FI parameter $r$ given by (6.2). For $\vartheta < -1/e^2\zeta$ we have $r < 0$. From the D-term (5.5) we see that the moduli space of vortices in this theory coincides with the moduli space of vortices in a U($M$) gauge theory, also with $N_f = (N + M)$ flavours. (We must also perform a parity transformation in going from the U($N$) theory to the U($M$) theory, so that $Z \to Z^\dagger$). Denote the gauge coupling of the U($M$) theory as $e'^2$, and the non-commutivity parameter as $\vartheta'$. Then for the moduli spaces of vortices to coincide, we simply require

$$\frac{1}{e'^2} + \vartheta'\zeta = -\frac{1}{e^2} - \vartheta\zeta \, .$$

This duality is, like many, reminiscent of the $(N_c = N) \to (N_f - N_c = M)$ duality of Seiberg.

## 7. Summary and discussion

Let us begin this ending with a summary of our results. We have studied vortices in U($N$) Yang-Mills theories with $N_f \geq N$ flavours. These theories have a FI parameter $\zeta$ which ensures the gauge group is completely broken in the vacuum. The gauge coupling parameter is $e^2$ and the theories may be defined on the non-commutative plane with $[x^1, x^2] = -i\vartheta$.

We have shown that the moduli space of charge $k$ vortices in this theory is described by the Higgs branch of a U($k$) gauge theory with four supercharges, coupled to $N$ chiral multiplets $\psi$ transforming in the $k$ representation of the gauge group, $N_f - N$ chiral multiplets $\tilde{\psi}$ transforming in the $\bar{k}$ representation, and a single chiral multiplet $Z$ transforming in the adjoint representation. This Higgs branch is defined by a U($k$) symplectic quotient of $\mathbf{C}^{k(k+N_f)}$ with moment map,

$$[Z^\dagger, Z] + \psi\psi^\dagger - \tilde{\psi}^\dagger\tilde{\psi} = r \, , \tag{7.1}$$

where the level of the moment map $r$ is a FI parameter defined by

$$r = 2\pi \left( \frac{1}{e^2} + \vartheta\zeta \right).$$

We further showed that the vortex moduli space may be constructed as a complex submanifold (or, for $N_f = N$, a special lagrangian submanifold) of the moduli space of $k$ instantons in non-commutative $\mathrm{U}(N_f)$ Yang-Mills theory.

Relationships between the instanton and vortex equations have been noted in the past. In particular, a reduction of instantons in SU(2) Yang-Mills on $\mathbb{R}^2 \times \mathbf{S}^2$ gives rise to the U(1) vortex equations [45]. While this relationship appears to share several characteristics of our correspondence, it differs in many important details. It would be interesting to elucidate the connections between these two approaches.

As we discussed in detail, the Manton metric on the moduli space of vortices does not coincide with the metric induced on the Higgs branch (7.1) by the Kähler quotient construction. Indeed, all our experience with gauge theories with four supercharges suggests that it is too ambitious an enterprise to determine the metric on the moduli space of vortices. Other, more topological, questions can be asked with greater success and the construction (7.1) provides the answers. As stressed in the introduction, these topological questions include certain quantum correlation functions in supersymmetric gauge theories.

Given the relationship described in section 4 between the moduli space of vortices and the moduli space of Yang-Mills instantons, one may expect quantitative agreement between topological correlation functions of two and four dimensional gauge theories. These would receive non-perturbative instanton corrections in four dimensions, and vortex corrections in two dimensions. Indeed, it is well known that $\mathcal{N} = 1$ $\mathrm{SU}(M+1)$ super-Yang-Mills in four dimensions shares many features with the $\mathcal{N} = (2,2)$ $\mathbf{CP}^M$ sigma-model in two dimensions including, most pertinently, its low-energy effective action [46, 47]. A second, more quantitative, example was given in [48] where $\mathcal{N} = 2$ theories in four dimensions were shown to have a spectrum of monopoles that coincides with the spectrum of kinks in certain $\mathcal{N} = (2,2)$ theories in two dimensions. Both monopole and kink masses receive identical corrections from non-perturbative effects, namely Yang-Mills instantons and vortices respectively. It seems likely that the semi-classical reason for these agreements can be traced to the relationship between the vortex and instanton moduli spaces described here.

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